Multiplicity of ordered phases in frustrated systems obtained from hard-spin mean-field theory

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Random quenched dilution of the triangular-lattice antiferromagnetic Ising model locally relieves frustration, leading to ordering phenomena. We have studied this system, under such dilution of one sublattice, using hard-spin mean-field theory. After a threshold dilution, two sublattices develop nonzero magnetizations of equal magnitude and opposite signs, as all three sublattices exhibit spin-glass order. In this phase, multiple sets of ordered solutions occur. A phase diagram is obtained in dilution fraction and temperature.

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A ''rugged free-energy landscape'' is often mentioned as a distinctive characteristic in the discussions of spin-glass systems [1]. Concrete support for such a phenomenon would derive from multiple solutions, not related to each other by a global symmetry, of self-consistent order-parameter equations. This has not been previously obtained for any system with a realistic spatial connectivity. In this work, we do find such multiple solutions, not related by global symmetry, in a random frustrated system with realistic spatial connectivity, namely, the quench-diluted triangular-lattice antiferromagnetic Ising model, studied via the closed-form implementation of hard-spin mean-field theory [2–12].

The antiferromagnetic Ising model, with Hamiltonian

$$-\beta\mathcal{H}=-J\sum_{\langle ij\rangle}s_i s_j, \quad J\geqslant 0, \tag{1}$$

where $s_i=\pm 1$ at each site i of a triangular lattice and $\langle ij\rangle$ indicates summation over nearest-neighbor pairs of sites, is fully frustrated [13]: In each elementary triangle, one of the three nearest-neighbor antiferromagnetic interactions is dissatisfied when the energy is minimized. This leads, macroscopically, to a highly degenerate system that is disordered at all nonzero $(1/J>0)$ temperatures [14]. Random quenched dilution of the system relieves the frustration at random localities and can be expected to lead to ordering phenomena. Indeed, a Monte Carlo study with random quenched dilution of all three sublattices equivalently has indicated spin-glass order [15]. We consider the random quenched dilution of the sites of one of three sublattices. After a threshold dilution, the system exhibits uniform and opposite magnetizations in two sublattices and spin-glass order, i.e., spins frozen in random directions, in the quench-diluted sublattice. A phase diagram is obtained in the variables of dilution fraction and temperature. Within the ordered phase, for a fixed dilution fraction and temperature, a multiplicity of solutions, distinguished by different values of the local and global order parameters, is obtained to the hard-spin mean-field equations.

We use hard-spin mean-field theory, a method which is almost as simply enunciated as usual mean-field theory, but which conserves frustration [2–12]. Consequently, this method has yielded, for example, the lack of order in the undiluted triangular-lattice antiferromagnetic Ising model and the ordering that occurs when a uniform magnetic field is applied to the system, in a quantitatively accurate phase diagram in the temperature versus magnetic field variables [2–5,8,9]. Hard-spin mean-field theory also yields the lack of order in the one-dimensional Ising ferromagnet and the occurrence of order in the two-dimensional Ising ferromagnet, the latter with an onset temperature improved over usual mean-field theory [4]. Hard-spin mean-field theory has also been successfully applied to complicated systems that exhibit a variety of ordering behaviors, such as three-dimensional stacked frustrated systems [2,6] and higher-spin systems [7].

The self-consistent equation for local magnetizations in hard-spin mean-field theory is

$$m_i=\sum_{\{s\}}\left[\prod_j p(m_j;s_j)\right]\tanh\left(-J\sum_{j'}s_{j'}\right), \tag{2}$$

where the product over j and sum over j' run over all nondiluted sites neighboring site i, and the single-site spin probability distribution $p(m_j;s_j)$ is $(1+m_j s_j)/2$. The outer summation is over the ± 1 values of the spins at the undiluted sites neighboring site i. Thus, the spin at each site feels the anti-aligning field due to the full (i.e., hard) spin of each of its neighbors. Equation (2) is a set of coupled equations for all the local magnetizations and is solved iteratively for a given realization of dilution in a finite but large system.

Alternatively, *a further approximation* is to impose sublatticewise uniformity, $m_j=m_\alpha$ for each sublattice $\alpha=a,b,c$, and to average the self-consistent equation over all realizations of quenched site dilution,

$$m_i=\sum_{\{\eta\}}Q(\{\eta\})\left\{\sum_{\{s\}}\left[\prod_j p(m_j;s_j)\right]\tanh\left(-J\sum_{j'}s_{j'}\right)\right\}. \tag{3}$$

In Eq. (3), the parentheses enclose the right-hand side of Eq. (2). This is summed over the 2^6 possible quenched environ

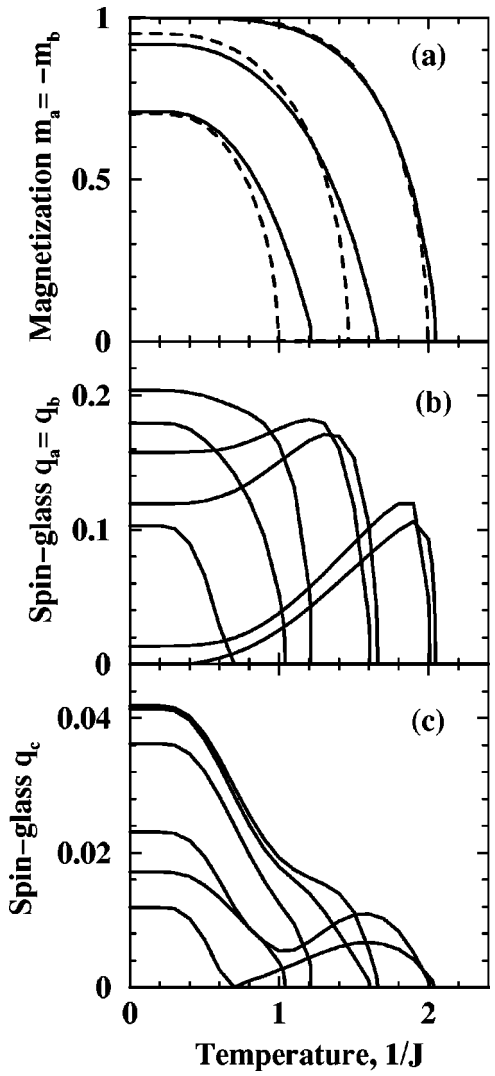

FIG. 1. Finite-temperature order in the random quench-diluted triangular-lattice antiferromagnetic Ising model: (a) Magnetizations of the two undiluted sublattices from hard-spin mean-field theory (solid curves). The three curves correspond to $p = 0.75, 0.5, 0.093\,75$, which can be identified by the respectively increasing critical temperatures, i.e., increasing x-axis intercepts. The result from the further approximation of Eq. (3) is given with the dashed curve. The quench-diluted sublattice has zero magnetization. (b) Spin-glass order parameter of the undiluted sublattices. The curves, again distinguished by their respectively increasing critical temperatures, are for $p = 0.9375, 0.890\,625, 0.75, 0.625, 0.5, 0.140\,625, 0.093\,75$. Note the reentrant behavior in the magnitudes, for high dilutions. (c) Spin-glass order parameter of the quench-diluted sublattice. The values of p are as in (b). Here, the spin-glass order for $p = 0.093\,75$ is away from zero only at higher temperatures. The spin-glass order for $p = 0.140\,625$ exhibits doubly reentrant behavior. The solid curves are for a fixed realization of the quenched disorder in a 24×24 system.

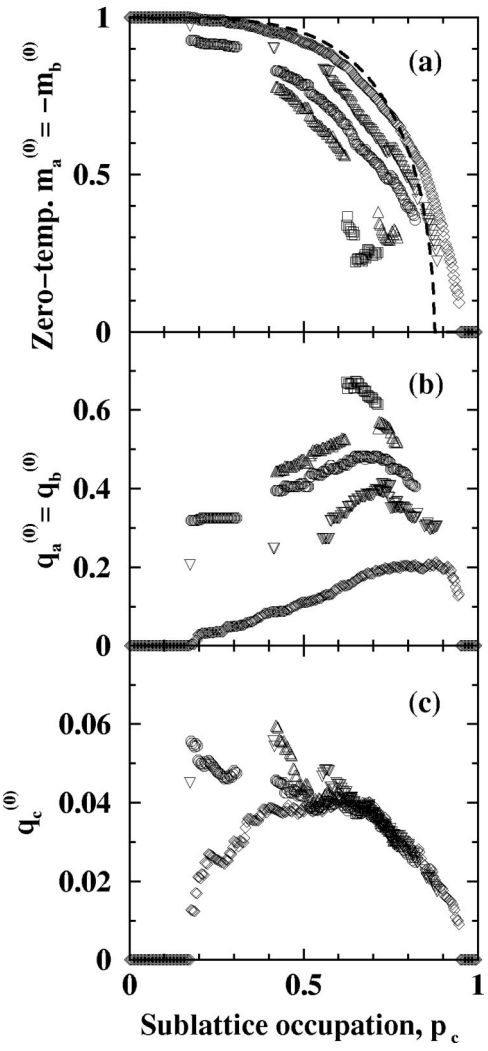

FIG. 2. Zero-temperature sublattice magnetizations and spin-glass order parameters from hard-spin mean-field theory. The sublattice magnetizations (a) do not saturate at zero-temperature, except for the full dilution (hexagonal lattice) limit. The dashed curve in (a) is the result of the further approximation of Eq. (3). The zero-temperature spin-glass order [(b) and (c)] is maximal at intermediate dilutions. Multiple solutions of the hard-spin mean-field theory equations are seen, in addition to the most stable set in depicted in Fig. 1. (For each solution, the same symbol is used in Figs. 2–6). This figure is obtained for $1/J = 0.0001$, in a 30×30 system.

ments $\{\eta\}$ of site i. Each quenched environment $\{\eta\}$ occurs with a probability Q composed of six factors q_j, with $q_j = (1 - p_j)$ for each quench-diluted neighbor j and $q_j = p_j$ for each undiluted neighbor j. Equation (3) can be compactly rewritten as

$$m_i = \left[\prod_j \sum_{\eta_j = 0,1} q(p_j; \eta_j) \sum_{s_j = \pm 1} p(m_j; s_j) \right] \times \tanh\left(-J \sum_{j'} \eta_{j'} s_{j'} \right), \tag{4}$$

where the product over j and sum over j' run over all sites neighboring site i, the single-site quenched-dilution probability distribution $q(p_j; \eta_j)$ is $1 - p_j - \eta_j(1 - 2p_j)$, and $p_j = p_\alpha$ for each sublattice $\alpha = a, b, c$. Equation (3) or, equivalently, Eq. (4) is solved for (m_a, m_b, m_c) for given (p_a, p_b, p_c).

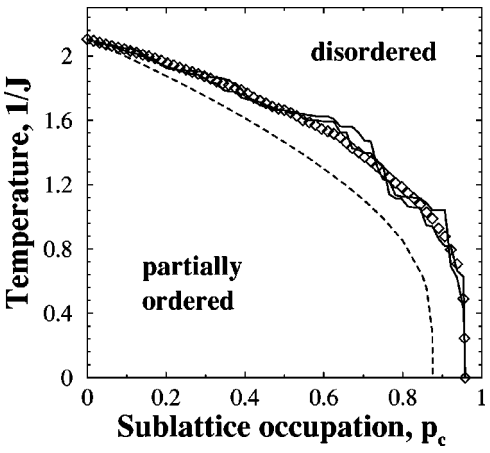

FIG. 3. Phase diagram of the random quench-diluted triangular-lattice antiferromagnetic Ising model. The solid curves show the phase boundaries obtained by hard-spin mean-field theory for two realizations of quenched dilutions of a 24×24 system. The losanges show the result from averaging over 15 such realizations. The dashed curve is the result from the further approximation of Eq. (3); this curve is given analytically by Eq. (7).

Whereas hard-spin mean-field theory [Eq. (2)] yields the variations in the local magnetizations within each sublattice due to differently quenched local environments, the implementation of Eq. (3) is a further approximation over hard-spin mean-field theory: While still incorporating frustration, it imposes sublatticewise uniform magnetizations. Equation (3) is a set of three coupled equations, whereas Eq. (2) is a set of N coupled equations, where N is the size of the system.

Upon random quenched dilution of one sublattice, the frustrated triangular-lattice Ising model does indeed show long-range order, as for example depicted in Figs. 1. The two undiluted sublattices (labeled a and b), which are now subject to random unfrustrated localities at the dilution points of the other sublattice (labeled c), develop nonzero sublattice-averaged magnetizations, $m_a=-m_b$, at low temperatures. For low dilutions, these magnetizations [Fig. 1(a)] show an initial slow growth at onset as temperature is lowered and do not saturate at zero temperature. The diluted sublattice c is also subject to local liftings of frustration, due to the spatially nonuniform magnetizations of sublattices a and b, but this is a secondary and, therefore, weaker effect, and sublattice c does not develop a nonzero sublattice-averaged magnetization, $m_c=0$.

All three sublattices develop, within the ordered phase, nonzero spin-glass order [16], i.e., randomly frozen order, with order parameters

$$q_\alpha=\left[\frac{1}{N_\alpha}\sum_i^\alpha (m_i-m_\alpha)^2\right]^{1/2}, \tag{5}$$

where N_α is the number of spins of sublattice α. Note that, for the quenched-diluted sublattice,

$$q_c=\left[\frac{1}{N_c}\sum_i^c m_i^2\right]^{1/2}. \tag{6}$$

At high dilutions, the spin-glass ordering trend shows reentrance (as temperature is lowered, increases and then decreases) on the undiluted sublattices [Fig. 1(b)] and double reentrance (increase, decrease, and again increase) on the diluted sublattice [Fig. 1(c)]. Maximal zero-temperature spin-glass order occurs at intermediate dilutions, as seen in Figs. 2(b) and 2(c).

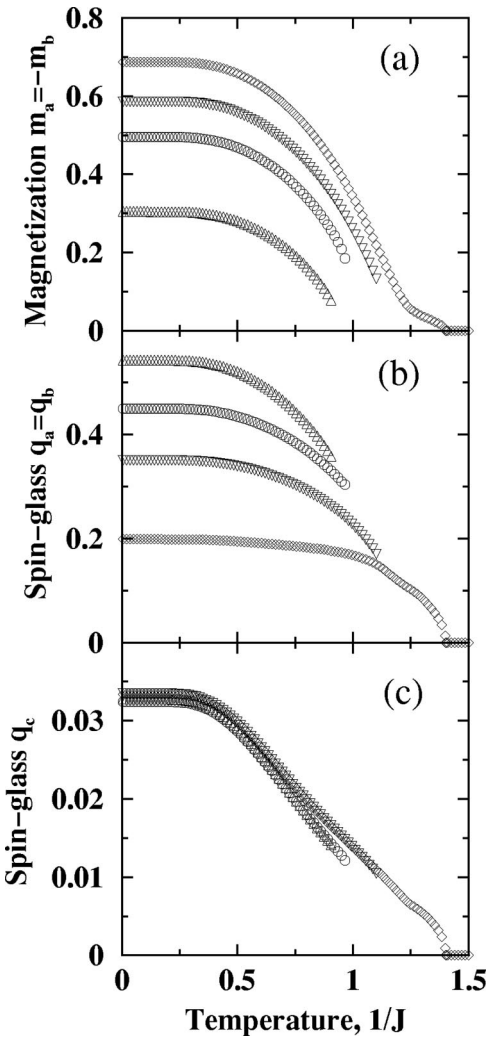

FIG. 4. Finite-temperature multiplicity of solutions for $p=0.75$, in the 30×30 system.

The phase diagram of the system is shown in Fig. 3. It is seen that a threshold dilution of 0.042 of one sublattice is needed for ordering, i.e., the occupancy p has to be below 0.958 for ordering. Figure 3 shows the phase boundaries obtained for two realizations of quenched dilutions of a 24×24 system and the result from averaging over 15 such realizations. Also shown by the dashed line in Fig. 3 is the further approximation of Eq. (3). This dashed phase boundary obeys the equation

$$p^3f_3+3p^2(1-p)f_2+3p(1-p)^2f_1+(1-p)^3f_0=4/3, \tag{7}$$

where

$$f_3=(t_6+6t_4+5t_2)/8, \quad f_2=(t_5+3t_3+2t_1)/4,$$

$$f_1=(t_4+2t_2)/2, \quad f_0=t_3+t_1,$$

where $t_n\equiv\tanh(nJ)$, and gives a dilution threshold of 0.125.

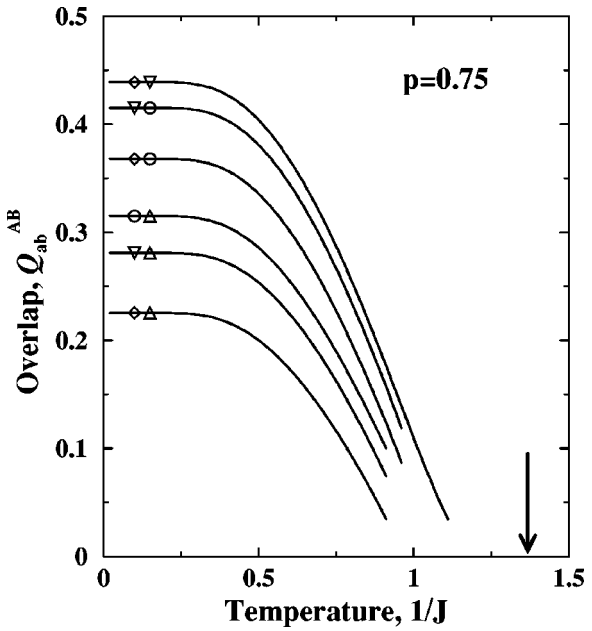

FIG. 5. Overlap Q between the different solutions [see Eq. (8a)] on the undiluted sublattices, for $p=0.75$. The arrow indicates the onset of order.

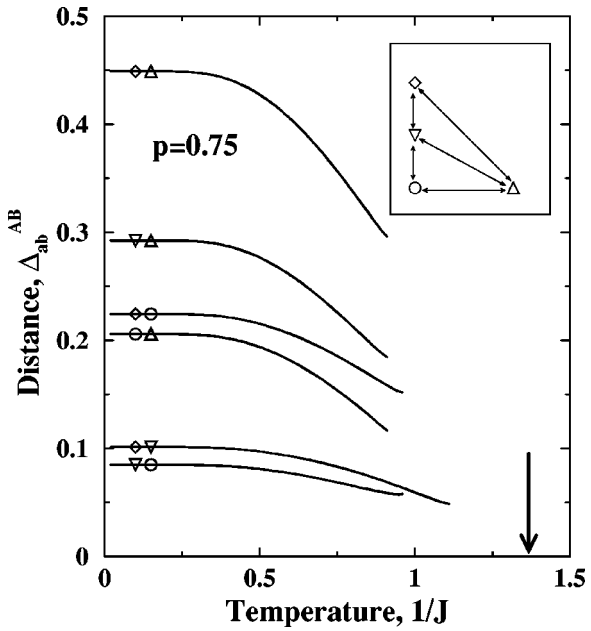

FIG. 6. Distance Δ between the different solutions [see Eq. (8b)] on the undiluted sublattices, for $p=0.75$. The arrow indicates the onset of order. The inset schematizes the separations between the different solutions, showing no ultrametricity.

Within the ordered phase, we find that the hard-spin mean-field equations (2) admit, as seen in Figs. 2 and 4, a multiplicity of solutions, in addition to the set depicted in Fig. 1. The latter is the most stable solution, in the sense that it has the largest basin of attraction under the iterative solution of Eq. (2). The other solutions appear at different temperatures below the onset temperature for the most stable solution. Since, in this study, the number of the sets of solutions increased in going from the 24×24 system to the 30×30 system (depicted in Figs. 2 and 4), it can be inferred that the solutions become numerous in the infinite system.

Figures 5 and 6 depict the overlaps between pairs of solutions (A,B),

$$Q_{\alpha}^{AB}=\frac{1}{N_{\alpha}}\sum_{i}^{\alpha} m_i^A m_i^B \tag{8a}$$

and the distances

$$\Delta_{\alpha}^{AB}=\frac{1}{N_{\alpha}}\sum_{i}^{\alpha} (m_i^A-m_i^B)^2. \tag{8b}$$

From Δ_{α}^{AB} in Fig. 6, we can deduce the separation, in local-order-parameter space, between the different solutions. This is shown schematically in the inset of the figure. We note that the different solutions are not ultrametrically [17] related, since no isosceles-triangle relations are seen. Thus, it may well be that ultrametricity is a property specific to the infinitely connected lattice.

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